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Approximation by repeated Padé approximants

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Abstract

It is well known that the nth diagonal Padé approximant (PA) of a Markov–Stieltjes function f converges with geometric rate to f. A special case of what we prove is that if $n = 2k$ the same rate of convergence may be obtained by first constructing the kth diagonal PA p_1/q_1 of f and then the kth diagonal PA of $q_1^2 f$.

Keywords: Approximation theory; Padé approximation; Markov–Stieltjes functions; Convergence results; Orthogonal polynomials

1. Introduction

Let f be a Markov–Stieltjes function,

$$f(z) = \int_{\Delta} \frac{\mathrm{d}\mu(t)}{z - t}, \tag{1}$$

where Δ is a compact interval on $\mathbb{R}$ and μ a finite positive measure whose support is an infinite subset of Δ. Then f is analytic in $\bar{\mathbb{C}} \setminus \Delta$ where $\bar{\mathbb{C}}$ denotes the extended complex plane, and $f(\infty) = 0$. Let $\pi_n(z)$ denote the nth diagonal Padé approximant (PA) at infinity of the function $f(z)$ (see Section 2 for the definition; for notational convenience we shall refer to the nth diagonal PA as the PA of order n). This means that

$$f(z) - \pi_n(z) = \mathcal{O}(z^{-2n-1}), \quad \text{as } z \to \infty,$$

where the right-hand side denotes a power series in $1/z$ with the lowest order term of degree $2n + 1$.

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Let $\phi(z)$ be a conformal mapping of the domain $\bar{\mathbb{C}}\setminus\Delta$ onto the complement of the unit disk $\{z: |z| \leqslant 1\}$ with $\phi(\infty) = \infty$. Then it is well-known (see [4, p. 63]) that we have the following geometric rate of convergence,

$$\limsup_{n\to\infty} |f(z) - \pi_n(z)|^{1/n} \leqslant \frac{1}{|\phi(z)|^2} < 1,$$

locally uniformly (on compact subsets) in $\bar{\mathbb{C}}\setminus\Delta$.

The object in this note is to prove that the same rate of convergence may be obtained in another way; we can divide our construction of an approximant into two steps. We assume for simplicity that $n = 2k$. First we construct the PA $p_1(z)/q_1(z)$ of order k of $f(z)$ and then the PA of the same order k of the function $q_1^2(z)f(z)$. (Note that in general $q_1^2(z)f(z)$ has a pole at infinity which means that we have to define PAs also for functions with a pole at infinity.) Consequently, instead of constructing immediately the PA of order $2k$, we divide our construction into two steps and in each step we construct PAs of order k. The two methods lead to different approximants but they give the same rate of convergence (see Theorem 1 below).

There are some reasons why the second method may be preferable. First, suppose that we have constructed the PA $p_1(z)/q_1(z)$ of order k of $f(z)$ and that $p_1(z)/q_1(z)$ approximates $f(z)$ within accuracy ε on some fixed compact set but that we need an accuracy, let us say, within ε^2. Then we can either construct the PA of order $2k$ of $f(z)$ or the PA of order k of $q_1^2(z)f(z)$. The latter way seems to be preferable because in this case we do not loose the information obtained in the first step (we use $q_1(z)$).

Second, if we construct immediately the PA of order $2k$, we have to solve a linear system of equations with matrix of order $2k \times 2k$ in order to find the denominator of degree $2k$ of the PA. To do this we need $(\frac{2}{3}k)^3 = 8(\frac{1}{3}k)^3$ operations (see [1, p. 63]). In the other case we have to solve two linear systems with $k \times k$ matrices. To do that we need only $2\,(\frac{1}{3}k)^3$ operations. Moreover, in the second case the construction may be divided not only into two but into several, say, m steps (see Theorem 2). Then the difference between the sizes of the matrices becomes more essential; an $mk \times mk$ matrix in the first case giving $(\frac{1}{3}mk)^3$ operations, and m matrices of order $k \times k$ in the second case giving $m\,(\frac{1}{3}k)^3$ operations.

2. Statement of results

We first define the PA at infinity. The definition below includes, of course, the case when f is analytic at infinity, for instance when f is of the form (1).

Definition. Let f be a function which is holomorphic in a punctured neighbourhood of infinity and has a removable singularity or a pole at infinity. The *Padé approximant* (*PA*) of order n of $f(z)$, or the nth diagonal PA of $f(z)$, is a rational function $p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, $q(z) \not\equiv 0$, and the degree of $q(z)$, $\deg q(z)$, is at most n, for which the difference $f(z) - P(z)/Q(z)$ has a zero of maximal order at infinity, if $P(z)$ and $Q(z)$ are polynomials, $Q(z) \not\equiv 0$, and $\deg Q(z) \leqslant n$.

It can be proved that the fraction $p(z)/q(z)$ is unique.

Below we will need an equivalent definition of the PA as the fraction $p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, $q(z) \not\equiv 0$, and $\deg q(z) \leqslant n$, for which $q(z)f(z) - p(z)$ has a zero of maximal order at infinity. Note that $p(z)$ is the polynomial part of $q(z)f(z)$.

Remark. Another equivalent definition of the PA $p(z)/q(z)$ of order n of $f(z)$ is as follows: $p(z)$ and $q(z)$, $q(z) \not\equiv 0$, are polynomials, and $\deg q(z) \leqslant n$, such that

$$q(z)f(z) - p(z) = \mathcal{O}(z^{-n-1}), \quad \text{as } z \to \infty,$$

where the right-hand side denotes a power series in z^{-1} with the lowest order term of degree $n+1$ or higher. This means that the $n+1$ coefficients of $q(z)$ are determined by solving a system of n linear equations. These equations are obtained by expanding $q(z)f(z)$ in a Laurent series around infinity and requiring that the coefficients of z^{-j}, $1 \leqslant j \leqslant n$, are zero. After that $p(z)$ is determined as the polynomial part of this expansion of $q(z)f(z)$.

If f has a removable singularity at infinity, p has degree at most n and p/q is often referred to as the PA of f of type (n, n). This is the case usually studied and it justifies the name diagonal PA. For a function of the form (1) p has degree at most $n-1$ since $f(\infty) = 0$, and p/q is then often referred to as the PA of f of type $(n-1, n)$.

Theorem 1. *Let $p_1(z)/q_1(z)$ be the PA of order n_1 of the Markov–Stieltjes function $f(z)$ given by* (1). *Let $p_2(z)/q_2(z)$ be the PA of order n_2 of the function $q_1^2(z)f(z)$. Put $n = (n_1, n_2)$, $|n| = n_1 + n_2$, and $Q_n(z) = q_1(z)q_2(z)$. Then $\deg q_1(z) = n_1$, $\deg q_2(z) = n_2$, and we have the following error formula for $z \in \bar{\mathbb{C}} \setminus \Delta$,*

$$r_n(z) := f(z) - \frac{p_2(z)}{q_2(z)q_1^2(z)} = \frac{1}{Q_n^2(z)} \int \frac{Q_n^2(t)}{z-t} \, \mathrm{d}\mu(t) = \mathcal{O}(z^{-2|n|-1}).$$

Also,

$$\limsup_{|n| \to \infty} |r_n(z)|^{1/|n|} \leqslant \frac{1}{|\phi(z)|^2} < 1,$$

locally uniformly in $\bar{\mathbb{C}} \setminus \Delta$, where $\phi(z)$ was defined in the Introduction.

Remark. In the theorem we defined $p_2(z)/q_2(z)$ as the PA of order n_2 of the function $q_1^2(z)f(z)$. But this means that $p_2(z)/(q_2(z)q_1^2(z))$ is a Padé type approximant of $f(z)$ with preassigned poles at the zeros of $q_1^2(z)$ (see [3]). This is an example where Padé type approximants appear in a natural way.

Theorem 1 has the following generalization.

Theorem 2. *Put $n = (n_1, n_2, \ldots, n_m)$ and let $p_j(z)/q_j(z)$, $1 \leqslant j \leqslant m$, denote the PA of order n_j of the function $[q_{j-1}(z) \cdot q_{j-2}(z) \cdots q_1(z)]^2 \cdot f(z)$. Put $|n| = n_1 + n_2 + \cdots + n_m$ and $Q_n(z) = q_1(z)q_2(z) \cdots q_m(z)$. Then $\deg q_j(z) = n_j$, $1 \leqslant j \leqslant m$, and the following error formula holds for $z \in \bar{\mathbb{C}} \setminus \Delta$,*

$$r_n(z) := f(z) - \frac{p_m(z)}{q_m(z)[q_{m-1}(z) \cdot q_{m-2}(z) \cdots q_1(z)]^2}$$
$$= \frac{1}{Q_n^2(z)} \int \frac{Q_n^2(t)}{z-t} \, \mathrm{d}\mu(t) = \mathcal{O}(z^{-2|n|-1}).$$

Also,

$$\limsup_{|n|\to\infty} |r_n(z)|^{1/|n|} \leqslant \frac{1}{|\phi(z)|^2} < 1,$$

locally uniformly in $\bar{\mathbb{C}}\setminus\Delta$.

In the proof of these theorems we shall use the following lemma which follows from [2, Ch. III, formula (7.8)].

Lemma 3. *Let* Δ *be a compact interval on the real line and* $\{q_n(z)\}_0^\infty$ *a sequence of polynomials where, for each* n, $q_n(z)$ *is an orthonormal polynomial of order* n *with respect to some probability measure* μ_n *on* Δ. *This means that*

$$\int_\Delta q_n(t)t^k \,\mathrm{d}\mu_n(t) = 0 \quad \textit{for } k = 0, 1, \ldots, n-1$$

and

$$\int_\Delta q_n^2(t)\,\mathrm{d}\mu_n(t) = 1.$$

Then

$$\limsup_{n\to\infty}\left|\frac{1}{q_n(z)}\right|^{1/n} \leqslant \frac{1}{|\phi(z)|}$$

locally uniformly in $\bar{\mathbb{C}}\setminus\Delta$.

3. Proofs

We shall prove Theorem 1. Theorem 2 is proved analogously.

It is well known that since $p_1(z)/q_1(z)$ is the Padé approximant of order n_1 of the Markov–Stieltjes function $f(z)$, $q_1(z)$ is an orthogonal polynomial of degree n_1 with respect to the measure μ (see for instance [3]). In particular, $\deg q_1(z) = n_1$ and the zeros of $q_1(z)$ are all real. Furthermore, we have

$$\begin{aligned} q_2(z)q_1^2(z)f(z) &= \int_\Delta \frac{q_2(z)q_1^2(z)}{z-t}\,\mathrm{d}\mu(t) \\ &= \int_\Delta \frac{q_2(z)q_1^2(z) - q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t) + \int_\Delta \frac{q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t) \\ &= p_2(z) + \int_\Delta \frac{q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t), \end{aligned}$$

where $p_2(z)$ is some polynomial. The last integral gives a function which is holomorphic at infinity with an expansion

$$\int_\Delta \frac{q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots,$$

where

$$c_k = \int_\Delta t^{k-1} q_2(t) q_1^2(t)\,\mathrm{d}\mu(t), \quad k = 1, 2, \ldots.$$

The denominator $q_2(z)$ of the PA $p_2(z)/q_2(z)$ of order n_2 is defined by the condition that $\deg q_2(z) \leqslant n_2$ and that the maximal number of the first coefficients c_k are zero. Since $q_1^2(t)\,\mathrm{d}\mu(t)$ is a positive measure on Δ we conclude that $q_2(z)$ must be an orthogonal polynomial of degree n_2 with respect to this measure. In particular, all the zeros of $q_2(z)$ are real. Hence, for the PA $p_2(z)/q_2(z)$ of order n_2 of $q_1^2(z)f(z)$ we have

$$q_1^2(z)f(z) - \frac{p_2(z)}{q_2(z)} = \frac{1}{q_2(z)} \int_\Delta \frac{q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t), \tag{2}$$

where $\deg q_2(z) = n_2$. Furthermore, since $q_2(z)$ is an orthogonal polynomial with respect to $q_1^2(t)\,\mathrm{d}\mu(t)$ we get

$$\begin{aligned} \frac{1}{q_2(z)} \int_\Delta \frac{q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t) &= \frac{1}{q_2^2(z)} \int_\Delta \frac{q_2(z)q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t) \\ &= \frac{1}{q_2^2(z)} \int_\Delta \frac{[(q_2(z)-q_2(t)) + q_2(t)]\,q_2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t) \\ &= \frac{1}{q_2^2(z)} \int_\Delta \frac{q_2^2(t)q_1^2(t)}{z-t}\,\mathrm{d}\mu(t). \end{aligned}$$

From this relation and (2) we get the error formula in Theorem 1. The estimation of the order of zero at infinity follows from the facts that $\deg q_1(z) = n_1$, $\deg q_2(z) = n_2$ and

$$\int_\Delta \frac{Q_n^2(t)}{z-t}\,\mathrm{d}\mu(t) = \mathcal{O}(z^{-1}) \quad \text{as } z \to \infty.$$

It remains to prove the limit relation in Theorem 1 from the error formula. It is straightforward to verify that it is enough to prove the limit relation for all sequences $n = (n_1, n_2)$ for which the following limits exist:

$$\lim_{|n|\to\infty} \frac{n_1}{|n|} = \alpha_1 \quad \text{and} \quad \lim_{|n|\to\infty} \frac{n_2}{|n|} = \alpha_2.$$

Clearly $\alpha_1 + \alpha_2 = 1$.

We may without loss of generality assume that $q_1(z)$ and $q_2(z)$ are normalized so that they are orthonormal polynomials. Then we get for any fixed compact $K \subset \bar{\mathbb{C}} \setminus \Delta$, and $z \in K$,

$$|r_n(z)| \leqslant \frac{1}{|(q_2 q_1)^2(z)|} \frac{1}{\operatorname{dist}(K, \Delta)} \int_\Delta q_2^2(t) q_1^2(t) \, d\mu(t)$$
$$= \frac{1}{\operatorname{dist}(K, \Delta)} \cdot \frac{1}{|(q_2 q_1)^2(z)|}, \tag{3}$$

where $\operatorname{dist}(K, \Delta)$ denotes the distance from K to Δ.

Furthermore, without loss of generality we may assume that μ is a probability measure. Then the polynomials $q_1(z) = q_{1,n}(z)$ with $\deg q_{1,n}(z) = n_1$ satisfy all the conditions of Lemma 3. Hence we have

$$\limsup_{|n| \to \infty} \left| \frac{1}{q_{1,n}(z)} \right|^{1/|n|} = \left[\limsup_{|n| \to \infty} \left| \frac{1}{q_{1,n}(z)} \right|^{1/n_1} \right]^{n_1/|n|} \leqslant \left| \frac{1}{\phi(z)} \right|^{\alpha_1}, \tag{4}$$

locally uniformly in $\bar{\mathbb{C}} \setminus \Delta$. Now we note that the measures $q_1^2 \, d\mu = q_{1,n}^2 \, d\mu$ are also probability measures on Δ due to the assumption of normality of $q_{1,n}$ with respect to μ. This means that the polynomials $q_2(z) = q_{2,n}(z)$ with $\deg q_{2,n}(z) = n_2$ also satisfy the conditions of Lemma 3. So analogously we have

$$\limsup_{|n| \to \infty} \left| \frac{1}{q_{2,n}(z)} \right|^{1/|n|} = \left[\limsup_{|n| \to \infty} \left| \frac{1}{q_{2,n}(z)} \right|^{1/n_2} \right]^{n_2/|n|} \leqslant \left| \frac{1}{\phi(z)} \right|^{\alpha_2}, \tag{5}$$

locally uniformly in $\bar{\mathbb{C}} \setminus \Delta$.

From (3)–(5) we get the desired limit relation and Theorem 1 is proved.

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